ON MONGE'S SOLUTION OF THE NON-INTEGRABLE EQUATION BETWEEN THREE VARIABLES.

By Prof. WILLIAM WOOLSEY JOHNSON, U. S. Naval Academy.

1. When the differential equation

$$Pdx + Qdy + Rdz = 0 \tag{1}$$

does not satisfy the condition of integrability,

$$P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) = 0,$$

its solution can only be expressed by means of two integral relations between x, y, and z. In fact, strictly speaking, a *particular* solution of (1) is in any case constituted by two equations and is geometrically represented by a line in space; the peculiarity of the integrable case being that all the lines representing particular integrals which pass through a given point A lie upon a definite surface passing through A. The equation of this surface (if it contain an arbitrary constant so that A may assume any position in space) will in that case constitute a *general* solution, or integral, each particular solution being made up of this integral and another equation which may be chosen in a perfectly arbitrary manner.

2. The geometrical distinction between the integrable and non-integrable case is usually illustrated by the consideration of an auxiliary system of lines; namely, the lines which satisfy the system of equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{2}$$

These lines constitute a doubly infinite system of lines in space, there being one line of the system, and only one, passing through any selected point A. Equation (1) then simply restricts a moving point to such paths as cut orthogonally at every point the auxiliary system (2). In the integrable case the auxiliary system of lines admits of a system of orthogonally cutting surfaces, and it is sufficient to say that the paths of the moving point (representing particular solutions of (1)) lie on the orthogonal surfaces, being otherwise unrestricted.

3. In the non-integrable case no such surfaces exist. We may first consider the restricted problem: "Assuming in advance one relation between x, y, and z, to determine the particular solutions of (1) which are consistent with the assumed relation." Geometrically, we thus assume a certain surface, and require the par-

ticular solutions of (1) which lie on this surface. At every point of the surface it is pierced by one of the lines of the auxiliary system, and there is in general but one direction upon the surface which is perpendicular to this line, and hence but one direction in which a point obeying equation (1) can move upon the assumed surface. Thus the solution of the restricted problem is represented by a singly infinite system of lines upon the surface determined by an additional integral equation containing one arbitrary constant.

4. When we attempt a general solution the equation of the assumed surface must contain an arbitrary function, but it is not necessary that it should be wholly arbitrary. For it is only essential to a general solution that it should include every particular solution. Consider now any particular solution of (1) represented by a line in space; it is only necessary that the assumed surface should be capable of containing this line. For example, it may be assumed as a cylindrical surface,

$$y = f(x),$$

whose elements are parallel to the axis of z, for such a surface can be passed through any given line. The additional equation which with $y = f(x)$ constitutes the solution will contain an arbitrary constant C, and any particular solution will be included in the general solution when we give a special form to the function f, and also a special value to the constant C.

5. Now the peculiarity of Monge's solution is due to the mode in which the arbitrary surface is formed, which is as follows: Let $V = C$ be an integral of the equation $Pdx + Qdy = 0$ when z is regarded as constant, and μ the corresponding integrating factor; then

$$V = \varphi(z) \tag{3}$$

is the assumed form of the arbitrary surface. The differential equation of this surface is then

$$\mu P dx + \mu Q dy + \left(\frac{dV}{dz} - \varphi'(z)\right) dz = 0, \tag{4}$$

which is identical with equation (1), if

$$\frac{dV}{dz} - \varphi'(z) = \mu R \tag{5}$$

identically; but this is only possible when

$$\frac{dV}{dz} - \mu R$$

happens to be a function of z only, which is only another mode of stating the

condition of integrability. But when φ in (3) is chosen arbitrarily, (5) is a relation between x, y, and z which together with (3) satisfies equation (1) and constitutes Monge's general solution. Thus the additional equation which constitutes the solution in this case is not found by integration, and contains no arbitrary constant.

6. It is to be noticed, however, that had the question been, as in section 3, that of determining the particular solutions of (1) which lie on the assumed surface (3) we should have had, as before, a singly infinite system of lines upon the surface. For, from equations (4) and (1) we can derive $dz = 0$, of which the integral is

$$z = c. \tag{6}$$

Thus upon the surface (3) we have the singly infinite system of solutions determined by (6); namely, the horizontal sections of the surface; and we have also the, so to say, singular solution determined by (5). It is now curious to note that when φ is made arbitrary, the general solution is made up not by the integral (6) which contains the arbitrary constant, but by the singular solution (5).

7. The geometrical interpretation of this fact is of interest. It is obvious from the mode in which the function V was determined that every horizontal section of the surface $V = \varphi(z)$ is a particular solution of (1). In fact $V = \varphi(z)$ is a surface passing through a consecutive series selected out of the doubly infinite system of lines which constitute the horizontal solutions, so to speak, of (1). In other words, it is a surface of which the horizontal elements are members of the system just mentioned. Consider now, as in section 4, the line in space representing any particular solution of (1) which we wish to show to be included in the general solution. The function φ can be so determined that the surface $V = \varphi(z)$ passes through the given line. Then the particular integrals which lie upon the surface $V = \varphi(z)$ are its horizontal sections determined by (6) and the singular solution determined by (5). Among these must be found the line representing a particular solution with which we started, and since this line is not in general capable of lying in a horizontal plane, it must be the singular solution corresponding to the surface determined. Thus every particular solution may be included in the solution consisting of equations (3) and (5) by giving a special form to the function φ; except the very restricted class of solutions which is included in the solution consisting of equations (3) and (6), or say of the equations $z = c$, $V = C$.

8. The relation of the singular solution to the auxiliary system of lines (2) is noteworthy. As mentioned in section 3 there is, in general, at a given point of the surface but one direction in which a point can move perpendicularly to the line of the auxiliary system which there pierces the surface. But the singular

solution is the locus of the points at which these lines pierce the surface orthogonally, so that a point may move along the singular solution and cut all the auxiliary lines it meets orthogonally.

9. The points considered in this paper may be illustrated by the following example: "Find the equation which must be associated with $x^2 + y^2 = \varphi(z)$ in order to give an integral of

$$[x(x-a) + y(y-b)]\,dz = (z-c)\,(xdx + ydy).\text{"}*$$

The assumed equation is in accordance with Monge's form, the value of μ being $2/(z-c)$. If φ is a given function, and we require all particular solutions which are consistent with the given equation, we must employ both the equations $z = C$ and

$$\varphi'(z) = \frac{2}{z-c}[x(x-a) + y(y-b)].$$

But if φ is arbitrary, the latter equation is that which must be associated with $x^2 + y^2 = \varphi(z)$ to include every solution of the differential equation *except* the solutions which are of the form

$$z = C, \qquad x^2 + y^2 = C'.$$

The auxiliary system (2) is in this case

$$\frac{dx}{x(z-c)} = \frac{dy}{y(z-c)} = -\frac{dz}{x(x-a) + y(y-b)},$$

of which the integrals are

$$y = \alpha x$$

and

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = \beta,$$

so that the auxiliary system of lines consists of all the circles in planes passing through the axis of z, having their centres at the feet of perpendiculars from (a, b, c) upon their planes. The form of the assumed surface,

$$x^2 + y^2 = \varphi(z),$$

is that of a surface of revolution having for its axis the axis of z, the horizontal circles which form the elements of such a surface cutting the auxiliary lines at right angles.

Now suppose for example we give to $\varphi(z)$ the form $k^2 - z^2$, so that the assumed surface becomes a sphere having its centre at the origin, then the lines upon this sphere which represent particular solutions of the given differential

* Forsyth's Differential Equations, Ex. 3, p. 252.

equation are the system of horizontal circles of the sphere, together with the single line which is its intersection with

$$-2z = \frac{2}{z-c}\left[x(x-a) + y(y-b)\right];$$

that is, with

$$ax + by + cz = k^2.$$

This last circle, the polar of the point (a, b, c) upon the sphere, is in fact readily seen to be the locus of the points at which the auxiliary lines pierce the sphere orthogonally.

10. With regard to the general solution of the equation

$$\left[x(x-a) + y(y-b)\right] dz = (z-c)(xdx + ydy),$$

it is also to be noticed that there is one particular integral which is of the nature of the solution in the integrable case; namely, $z = c$. The plane $z = c$, in fact, cuts the auxiliary system of lines orthogonally, and is the only surface which does so. Accordingly, the general solution consists of all curves in the plane $z = c$, the circles defined by

$$z = c,$$

$$x^2 + y^2 = C',$$

and the curves defined by

$$x^2 + y^2 = \varphi(z),$$

$$(z-c)\,\varphi'(z) = 2x(x-a) + 2y(y-b),$$

in which φ is arbitrary.